

# N=4 Supersymmetric MICZ-Kepler systems on $S^3$

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## Abstract

Proceeding from the superfield action for  $N = 4, d = 1$  nonlinear supermultiplet, equipped with the most general potential term, we find the action describing a charged particle on the sphere  $S^3$  in the field of  $n$  fixed Dirac dyons. We construct the supercharges and Hamiltonian and analyze some particularly interesting potentials corresponding to the  $N = 4$  supersymmetric extension of the integrable one- and two-center McIntosh–Cisneros–Zwanziger–Kepler (MICZ-Kepler) systems on  $S^3$ .

# 1 Introduction

The McIntosh–Cisneros–Zwanziger–Kepler (MICZ–Kepler) system is the integrable mechanical model which generalizes the Kepler (Coulomb) problem for the situation, when the conventional Coulomb center is replaced by the Dirac dyon, i.e. a particle carrying both electric and magnetic charges. The main feature of this system consists in the additional centrifugal potential  $U_{MICZ}(r) = \frac{s^2}{2mr^2}$  term<sup>1</sup> which appears in the Hamiltonian due to monopole-like nature of the forced center [1]. The properties of the MICZ–Kepler system are rather similar to the ordinary Coulomb one. For instance, beside the conserved angular momentum, the system has another integral of motion which is the perfect analog of the Laplace–Runge–Lenz vector. At the classical level, trajectories in the MICZ–Kepler system have the same shape as in the underlying Coulomb one, but in contrast to the latter case, the orbital plane is not always orthogonal to the angular momentum. Being quantized, the MICZ–Kepler system leads to the same spectrum as the Coulomb problem, with a little difference consisting in the shift of the possible values of the orbital quantum number – it starts with  $|s|$ . The Hamiltonian of MICZ–Kepler system which describes the motion of electrically charged scalar particle in the field of static Dirac dyon reads

$$\mathcal{H} = \frac{1}{2m} (\mathbf{p} - e\mathbf{A}_g)^2 - \frac{eq}{r} + \frac{s^2}{2mr^2}, \quad \text{rot}\mathbf{A}_g = \frac{g\mathbf{r}}{r^3}. \quad (1.1)$$

Obviously, there are many ways to construct the multi-center generalization of the Hamiltonian (1.1). Of course, the preferable generalization has to preserve the main property of the MICZ–Kepler system – its integrability. Quite interestingly,  $N = 4$  supersymmetry ruled out just the unique generalization of (1.1). It has been shown in [2, 3] that the proper multi-center generalization of the MICZ–Kepler system reads

$$\mathcal{H} = \frac{1}{2m} \left( \mathbf{p} - e \sum_{i=1}^n \mathbf{A}_{g_i}(\mathbf{r} - \mathbf{a}_i) \right)^2 - e \sum_{i=1}^n \frac{q_i}{|\mathbf{r} - \mathbf{a}_i|} + \frac{e^2}{2m} \left( \sum_{i=1}^n \frac{g_i}{|\mathbf{r} - \mathbf{a}_i|} \right)^2, \quad \text{rot}\mathbf{A}_{g_i}(\mathbf{r}) = \frac{g_i\mathbf{r}}{r^3}. \quad (1.2)$$

This Hamiltonian describes the motion of an electrically charged scalar particle in the field of  $n$  Dirac dyons sitting at the points with coordinates  $\mathbf{a}_i$ . Just with such structure of potential terms, the Hamiltonian admits  $N = 4$  supersymmetrization and, moreover, it describes a classically integrable system, at least for the two centers case.

One of the possible ways to further extend the system (1.2) is to consider the MICZ–Kepler system on the sphere  $S^3$  in the field of  $n$  Dirac dyons. Clearly, the  $N = 4$  supersymmetry, provided such a superextension exists, should help to find a proper multi-center extension. While trying to construct the  $N = 4$  supersymmetric version of the MICZ–Kepler system, one may immediately conclude that there are two possibilities to have a sphere  $S^3$  in the bosonic sector. Firstly, one may start with the  $N = 4, d = 1$  tensor supermultiplet [4], which contains on-shell three bosonic and four fermionic components. With a properly chosen metrics, one may get the sphere  $S^3$  in the bosonic sector. Then one may add the most general potential term, following the general construction [5, 6]. When the bosonic metric is completely fixed to be the  $S^3$  one, the possible potential terms are completely defined by a function obeying the flat three-dimensional Laplace equation. Clearly, in such a way it is impossible to get the monopole potential on  $S^3$ .

Alternatively, one may start with the  $N = 4, d = 1$  nonlinear supermultiplet [7, 8], which contains again three bosonic and four fermionic components on-shell. After fixing the metric, the potential terms are defined now by an arbitrary function obeying the three-dimensional Laplace equation on  $S^3$ . Just this case is what we are going to analyze in full details in the present work. In Section 2 we shortly describe the superspace construction of the corresponding Lagrangian and potential terms. In Section 3 we deal with the components approach. We present the Hamiltonian and supercharges for arbitrary potential terms. The main properties of these potentials is that they are fully determined by an arbitrary function which has to obey Laplace equation on the sphere  $S^3$ . In Section 4 we consider two particular cases of potential terms, i.e. with spherical and cylindrical symmetries, which seem to be the most interesting ones. Finally, we conclude with some comments.

## 2 $N = 4, d = 1$ nonlinear supermultiplet

The  $N = 4, d = 1$  nonlinear supermultiplet has been constructed in [7] and then further analyzed in [8]. It is defined in terms of the three  $N = 4, d = 1$  superfields  $\Phi, \Lambda, \bar{\Lambda}$  subject to the constraints:

$$\begin{aligned} D^1\Lambda &= -\Lambda D^2\Lambda, & \bar{D}_2\Lambda &= \Lambda \bar{D}_1\Lambda, & D^2\bar{\Lambda} &= \bar{\Lambda} D^1\bar{\Lambda}, & \bar{D}_1\bar{\Lambda} &= -\bar{\Lambda} \bar{D}_2\bar{\Lambda}, \\ iD^1\Phi &= -D^2\Lambda, & i\bar{D}_1\Phi &= \bar{D}_2\bar{\Lambda}, & iD^2\Phi &= -D^1\bar{\Lambda}, & i\bar{D}_2\Phi &= \bar{D}_1\Lambda, \end{aligned} \quad (2.1)$$

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<sup>1</sup> $s = eg$  is the so-called monopole number,  $e$  and  $m$  are the electric charge and the mass of the probe particle,  $g$  is the magnetic charge of dyon.

where spinor derivatives are defined by

$$D^i = \frac{\partial}{\partial \theta_i} + i\bar{\theta}^i \partial_t, \quad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} + i\theta_i \partial_t, \quad \{D^i, \bar{D}_j\} = 2i\delta_j^i \partial_t. \quad (2.2)$$

The constraints (2.1) leave in the nonlinear supermultiplet three physical  $\lambda, \bar{\lambda}, \phi$  and one auxiliary  $A$  bosonic fields and four fermionic fields  $\psi_a, \bar{\psi}^a$  ( $a = 1, 2$ ), which may be defined as

$$\begin{aligned} \phi &= \Phi|, \quad \lambda = \Lambda|, \quad \bar{\lambda} = \bar{\Lambda}|, \quad A = (D^1 \bar{D}_1 - \bar{D}_1 D^1) \Phi|, \\ \psi_1 &= \frac{1}{2} \bar{D}_1 \Phi|, \quad \psi_2 = -\frac{1}{2} \bar{D}_2 \Phi|, \quad \bar{\psi}^1 = -\frac{1}{2} D^1 \Phi|, \quad \bar{\psi}^2 = \frac{1}{2} D^2 \Phi| \end{aligned} \quad (2.3)$$

where  $|$  means  $\theta_i = \bar{\theta}^j = 0$ . The transformation properties of these components under  $N = 4$  supersymmetry read as follows:

$$\begin{aligned} \delta \lambda &= -2i(\epsilon_2 - \epsilon_1 \lambda) \bar{\psi}^1 + 2i(\bar{\epsilon}^1 + \lambda \bar{\epsilon}^2) \psi_2, \quad \delta \phi = 2(\epsilon_1 \bar{\psi}^1 - \epsilon_2 \bar{\psi}^2 - \bar{\epsilon}^1 \psi_1 + \bar{\epsilon}^2 \psi_2), \\ \delta \psi_1 &= -\frac{1}{2} \epsilon_1 \left( i\dot{\phi} + \frac{1}{2} A \right) - \frac{1}{2} \epsilon_2 \left( 2\dot{\lambda} + 4i\psi_1 \bar{\psi}^2 + i\bar{\lambda} \dot{\phi} + \frac{1}{2} \bar{\lambda} A \right), \\ \delta \psi_2 &= \frac{1}{2} \epsilon_2 \left( i\dot{\phi} - \frac{1}{2} A \right) + \frac{1}{2} \epsilon_1 \left( 2\dot{\lambda} - 4i\psi_2 \bar{\psi}^1 - i\lambda \dot{\phi} + \frac{1}{2} \lambda A \right), \\ \delta A &= -4i(\epsilon_1 \dot{\psi}^1 + \epsilon_2 \dot{\psi}^2 + \bar{\epsilon}^1 \dot{\psi}_1 + \bar{\epsilon}^2 \dot{\psi}_2). \end{aligned} \quad (2.4)$$

The general sigma-model type off-shell action has the form [7]

$$S = \int dt d\theta^2 d\bar{\theta}^2 L(\Phi, \Lambda, \bar{\Lambda}), \quad (2.5)$$

where  $L(\Phi, \Lambda, \bar{\Lambda})$  is an arbitrary real function of the superfields  $(\Phi, \Lambda, \bar{\Lambda})$ . The simplest potential term may be generated in a standard manner by adding to the action (2.5) the Fayet-Iliopoulos term

$$\tilde{S}_p = m \int dt A, \quad (2.6)$$

with  $m$  being the coupling constant. This potential term gives rise to the interaction with the electric field, but it will never produce the interaction with the magnetic field. Fortunately, for the nonlinear supermultiplet there is a more general Fayet-Iliopoulos term. Indeed, it has been shown in [8] that one may define the generalized auxiliary component  $B$  as

$$B = h_\phi A + b\dot{\lambda} + \bar{b}\dot{\bar{\lambda}} + a(\bar{\psi}^1 \psi_1 - \bar{\psi}^2 \psi_2) + a_1 \bar{\psi}^2 \psi_1 + a_2 \bar{\psi}^1 \psi_2, \quad (2.7)$$

where

$$\begin{aligned} a &= -8 \frac{h_{\phi\phi}}{1 + \lambda\bar{\lambda}}, \quad a_1 = -8ih_{\phi\bar{\lambda}} + 8\lambda \frac{h_{\phi\phi}}{1 + \lambda\bar{\lambda}}, \quad a_2 = 8ih_{\phi\lambda} + 8\bar{\lambda} \frac{h_{\phi\phi}}{1 + \lambda\bar{\lambda}}, \\ b &= 2ih_\lambda + 4\bar{\lambda} \frac{h_\phi}{1 + \lambda\bar{\lambda}}, \quad \bar{b} = -2ih_{\bar{\lambda}} + 4\lambda \frac{h_\phi}{1 + \lambda\bar{\lambda}}, \end{aligned} \quad (2.8)$$

and  $h$  obeys the Laplace equation on  $S^3$ :

$$h_{\phi\phi} + (1 + \lambda\bar{\lambda}) h_{\lambda\bar{\lambda}} + i\lambda h_{\lambda\phi} - i\bar{\lambda} h_{\bar{\lambda}\phi} = 0. \quad (2.9)$$

With all these equations (2.8),(2.9) being satisfied, the new auxiliary component (2.7) transforms under  $N = 4$  supersymmetry through a full time derivative [8]. Therefore, we may add to the action (2.5) a new generalized Fayet-Iliopoulos term:

$$\hat{S} = S + m \int dt B. \quad (2.10)$$

As we will see in the next Section, the action (2.10) provides the most general interaction with electric and magnetic fields.

To close this Section let us clarify in more details the differences between linear and nonlinear  $N = 4$  supermultiplets. For this purpose we will construct the most general potential term in (2.10) for both these supermultiplets in a different way. First of all let us rewrite the basic constraints (2.1) as follows

$$D^1\Lambda = i\alpha\Lambda D^1\Phi, \quad \overline{D}_1\overline{\Lambda} = -i\alpha\overline{\Lambda}\overline{D}_1\Phi, \quad (2.11)$$

$$D^2\Lambda = -iD^1\Phi, \quad \overline{D}_2\Lambda = \alpha\Lambda\overline{D}_1\Lambda, \quad iD^2\Phi = -D^1\overline{\Lambda}, \quad i\overline{D}_2\Phi = \overline{D}_1\Lambda, \quad (2.12)$$

$$D^2\overline{\Lambda} = \alpha\overline{\Lambda}D^1\overline{\Lambda}, \quad \overline{D}_2\overline{\Lambda} = i\overline{D}_1\Phi.$$

Here, we introduce the parameter  $\alpha$  to discuss two cases simultaneously: with  $\alpha = 0$  we have the standard linear  $N = 4$  tensor supermultiplet [4], while for the  $\alpha \neq 0$  one may always rescale the superfields to achieved  $\alpha = 1$  just as in the basic constraints (2.1). It is clear from (2.12) that the  $D^2$  and  $\overline{D}_2$  derivatives from all our superfields are expressed through  $D^1$  and  $\overline{D}_1$  derivatives from the same set of superfields. This means that all components of our (linear)nonlinear supermultiplet appear in the  $N = 2$  superfields  $\hat{\Lambda}, \hat{\overline{\Lambda}}, \hat{\Phi}$

$$\hat{\Lambda} = \Lambda_{\theta_2=\bar{\theta}^2=0}, \quad \hat{\overline{\Lambda}} = \overline{\Lambda}_{\theta_2=\bar{\theta}^2=0}, \quad \hat{\Phi} = \Phi_{\theta_2=\bar{\theta}^2=0}, \quad (2.13)$$

which depend only on  $\theta_1$  and  $\bar{\theta}^1$ . On these  $N = 2$  superfields the another implicit  $N = 2$  supersymmetry is realized as follows

$$\delta\hat{\Lambda} = i\epsilon_2 D^1\hat{\Phi} - \alpha\bar{\epsilon}^2\hat{\Lambda}\overline{D}_1\hat{\Lambda}, \quad \delta\hat{\overline{\Lambda}} = -\alpha\epsilon_2\hat{\overline{\Lambda}}D^1\hat{\overline{\Lambda}} - i\bar{\epsilon}^2\overline{D}_1\hat{\Phi}, \quad \delta\hat{\Phi} = -i\epsilon_2 D^1\hat{\overline{\Lambda}} + i\bar{\epsilon}^2\overline{D}_1\hat{\Lambda}. \quad (2.14)$$

Now, one may immediately write the most general potential term as

$$S_p = m \int dt d\theta_1 d\bar{\theta}^1 H(\hat{\Lambda}, \hat{\overline{\Lambda}}, \hat{\Phi}). \quad (2.15)$$

where, for the time being,  $H$  is an arbitrary function.

By construction, the potential term (2.15) is manifestly invariant with respect to  $N = 2$  supersymmetry realized on the  $(t, \theta_1, \bar{\theta}^1)$ . With respect to implicit  $N = 2$  supersymmetry (2.14) the integrand in (2.15) transforms as follows (we will write only  $\epsilon_2$  part of the variation)

$$\delta H = \epsilon_2 \left( H_{\hat{\Lambda}} \delta\hat{\Lambda} + H_{\hat{\overline{\Lambda}}} \delta\hat{\overline{\Lambda}} + H_{\hat{\Phi}} \delta\hat{\Phi} \right) = -\epsilon_2 \left[ -iH_{\hat{\Lambda}} D^1\hat{\Phi} + \left( iH_{\hat{\Phi}} + \alpha\hat{\overline{\Lambda}}H_{\hat{\overline{\Lambda}}} \right) D^1\hat{\overline{\Lambda}} \right]. \quad (2.16)$$

If we insist on the invariance of the potential term (2.15) under (2.16) the variation (2.16) must be represented as

$$\delta H = -\epsilon_2 D^1 G(\hat{\Lambda}, \hat{\overline{\Lambda}}) = -\epsilon_2 \left[ \left( G_{\hat{\Phi}} + i\alpha\hat{\Lambda}G_{\hat{\Lambda}} \right) D^1\hat{\Phi} + G_{\hat{\overline{\Lambda}}} D^1\hat{\overline{\Lambda}} \right], \quad (2.17)$$

where  $G(\hat{\Lambda}, \hat{\overline{\Lambda}}, \hat{\Phi})$  is an arbitrary function on its arguments and we used the constraints (2.11). Comparing (2.16) and (2.17) we will get the following conditions

$$iH_{\hat{\Phi}} + \alpha\hat{\overline{\Lambda}}H_{\hat{\overline{\Lambda}}} = G_{\hat{\overline{\Lambda}}}, \quad -iH_{\hat{\Lambda}} = G_{\hat{\Phi}} + i\alpha\hat{\Lambda}G_{\hat{\Lambda}}. \quad (2.18)$$

The integrability of the constraints (2.18) gives us the desired constraints on the super-potential  $H(\hat{\Lambda}, \hat{\overline{\Lambda}}, \hat{\Phi})$

$$\left( 1 + \alpha^2 \hat{\Lambda} \hat{\overline{\Lambda}} \right) H_{\hat{\Lambda}\hat{\overline{\Lambda}}} + H_{\hat{\Phi}\hat{\Phi}} + i\alpha \left( \hat{\Lambda} H_{\hat{\Phi}\hat{\Lambda}} - \hat{\overline{\Lambda}} H_{\hat{\Phi}\hat{\overline{\Lambda}}} \right) = 0. \quad (2.19)$$

Thus we conclude, the potential term (2.15) is invariant with respect to  $N = 4$  supersymmetry if its integrand obeys to the equation (2.19).

Now the differences between liner and nonlinear supermultiplet becomes transparent: the potential term for the nonlinear supermultiplet is defined by a harmonic on  $S^3$  super function, while for the linear tensor supermultiplet this function has to obey flat Laplace equation ( $\alpha = 0$ ). Being rewritten in the components, the potential term (2.15) is coincides with the potential in (2.10) after identification

$$H(\hat{\Lambda}, \hat{\overline{\Lambda}}, \hat{\Phi})|_{\theta_1=\bar{\theta}^1=0} = h(\lambda, \bar{\lambda}, \phi). \quad (2.20)$$

It is worth to note that the most general  $N = 4$  supersymmetric action for the conformally flat case has been constructed many years ago in [9]. We would like to stress again that while the kinetic parts in the  $N = 4$  actions for linear and nonlinear supermultiplet describe the conformally flat three-dimensional bosonic manifold, the structure of the potential terms is completely different in these cases. The main reason for this is the nonlinear realization of the off-shell supersymmetry on the components in the nonlinear case (2.4). This is the reason why the action (2.10) cannot be obtained within the approaches in [4], [9].

Moreover, in the next Section we will explicitly demonstrate that even the kinetic parts of the actions are different for the linear and nonlinear supermultiplets.

### 3 Components description: Lagrangian and Hamiltonian

In order to clarify the structure of the action (2.10), let us go to components. For doing this, one should perform an integration over Grassmann variables in (2.10) (with the constraints (2.1) imposed), and then eliminate the auxiliary component  $A$ . Before carrying out this task, let us make two essential comments.

First of all, we are interested to get a  $S^3$  sphere in the bosonic sector of the action. It has been shown in [7] that for this case the superfield Lagrangian  $L$  in (2.5) has to be chosen as

$$L = \ln(1 + \Lambda\bar{\Lambda}). \quad (3.1)$$

Secondly, after going to components, the kinetic terms for the fermions read

$$L_f = \frac{8i}{1 + \lambda\bar{\lambda}} \left[ \dot{\psi}_1 \bar{\psi}^1 + \dot{\psi}_2 \bar{\psi}^2 + \frac{1}{1 + \lambda\bar{\lambda}} \left( \dot{\lambda} \psi_1 \bar{\psi}^2 - \dot{\lambda} \psi_2 \bar{\psi}^1 - \lambda \dot{\bar{\lambda}} \psi_1 \bar{\psi}^1 - \bar{\lambda} \dot{\lambda} \psi_2 \bar{\psi}^2 \right) \right]. \quad (3.2)$$

One may easily check that this expression can be drastically simplified after passing to the new fermionic fields<sup>2</sup>

$$\psi = \frac{\bar{\psi}^2 + \bar{\lambda} \bar{\psi}^1}{1 + \lambda\bar{\lambda}}, \quad \xi = \frac{\bar{\psi}^1 - \lambda \bar{\psi}^2}{1 + \lambda\bar{\lambda}}, \quad (3.3)$$

in term of which it take the standard free form

$$L_f = -8i \left( \psi \dot{\bar{\psi}} + \xi \dot{\bar{\xi}} \right). \quad (3.4)$$

Taking all this into account, we may perform the integration over Grassmann variables and eliminate the auxiliary component  $A$ . After passing to the newly defined fermions (3.3), we end up with the following action:

$$\begin{aligned} S = \int dt & \left[ \frac{4\dot{\lambda}\dot{\bar{\lambda}}}{(1 + \lambda\bar{\lambda})^2} + \left( \dot{\phi} + i \frac{\dot{\lambda}\bar{\lambda} - \dot{\bar{\lambda}}\lambda}{1 + \lambda\bar{\lambda}} \right)^2 - m^2 h_\phi^2 + 2mh_\phi \frac{\partial_t(\lambda\bar{\lambda})}{1 + \lambda\bar{\lambda}} + 2im \left( h_\lambda \dot{\lambda} - h_{\bar{\lambda}} \dot{\bar{\lambda}} \right) \right. \\ & \left. - 8i \left( \psi \dot{\bar{\psi}} + \xi \dot{\bar{\xi}} \right) - 8m(1 + \lambda\bar{\lambda}) \left[ h_{\lambda\bar{\lambda}} (\bar{\xi}\xi - \bar{\psi}\psi) + (ih_{\phi\lambda} + \bar{\lambda}h_{\lambda\bar{\lambda}}) \bar{\psi}\xi + (-ih_{\phi\bar{\lambda}} + \lambda h_{\lambda\bar{\lambda}}) \bar{\xi}\psi \right] \right]. \end{aligned} \quad (3.5)$$

The bosonic kinetic terms of the action (3.5) describe just the sphere  $S^3$  in stereographic coordinates. What is a really interesting is that the  $N = 4$  supersymmetrization of this  $S^3$  can be achieved by adding four free fermions. Let us remind, that just the same phenomenon appears in the case of the  $N = 4$  supersymmetrization of the sphere  $S^2$  [7]. In addition, in the action (3.5) there are potential terms which are completely specified by the function  $h$  obeying Laplace equation on  $S^3$  (2.9).

Before going to the construction of the Hamiltonian and supercharges, let us note that the kinetic part of the action (3.5) can be brought into the simpler form

$$S_{kin} = \int dt \left( \frac{4(\dot{\mathbf{x}}\dot{\mathbf{x}})}{(1 + \mathbf{x}^2)^2} - 8i \left( \psi \dot{\bar{\psi}} + \xi \dot{\bar{\xi}} \right) \right), \quad (3.6)$$

where the new coordinates  $\mathbf{x} = (x_1, x_2, x_3)$  are related with the initial ones as

$$\lambda = \frac{2x_3 + i(1 - \mathbf{x}^2)}{2(x_1 + ix_2)}, \quad \bar{\lambda} = \frac{2x_3 - i(1 - \mathbf{x}^2)}{2(x_1 - ix_2)}, \quad e^{i\phi} = -\frac{x_1 - ix_2}{x_1 + ix_2}. \quad (3.7)$$

The action (3.6) yields a perfect opportunity to further clarify the differences between linear and nonlinear supermultiplets. From the paper [9] we know that the  $N = 4$  supersymmetric action with linear supermultiplet has the four-fermionic term

$$S \sim \int dt \left[ G \dot{\mathbf{v}} \dot{\mathbf{v}} - \left( \triangle G - \frac{\partial_m G \partial_m G}{2G} \right) \psi \bar{\psi} \xi \bar{\xi} + L_{fer} \right]. \quad (3.8)$$

Here,  $G(v^m)$ ,  $m = 1, 2, 3$  is an arbitrary metric and  $L_{fer}$  stands for the terms which are quadratic in fermions. Clearly, for the sphere  $S^3$  this four fermionic term unavoidably appears in the action. In the same time, the action (3.6), being  $N = 4$  supersymmetric, does not contain such term. Thus, the same bosonic manifold, the sphere  $S^3$  in our explicit example, can be supersymmetrized in two different ways. The reason is the existence of two *different*

<sup>2</sup>The same transformations have been used in [7] for the case of a particle on  $S^2$ .

off-shell realizations of  $N = 4$  supersymmetry on the three physical bosons, four fermions and one auxiliary field. Thus, the  $N = 4$  mechanics we are considering here is different from those one constructed in [9].

Due to the extremely simple structure of the action (3.5), the construction of the Hamiltonian does not contain any peculiarities. As usual, one should define the momenta  $p_\lambda, p_{\bar{\lambda}}, p_\phi, \pi_\psi, \pi_\xi$

$$\begin{aligned} p_\lambda &= \frac{4\dot{\bar{\lambda}}}{(1+\lambda\bar{\lambda})^2} + 2i\frac{\bar{\lambda}}{1+\lambda\bar{\lambda}} \left( \dot{\phi} + i\frac{\dot{\lambda}\bar{\lambda} - \lambda\dot{\bar{\lambda}}}{1+\lambda\bar{\lambda}} \right) + 2imh_\lambda + 2mh_\phi \frac{\bar{\lambda}}{1+\lambda\bar{\lambda}}, \\ p_\phi &= 2 \left( \dot{\phi} + i\frac{\dot{\lambda}\bar{\lambda} - \lambda\dot{\bar{\lambda}}}{1+\lambda\bar{\lambda}} \right), \quad \pi_\psi = 4i\bar{\psi}, \quad \pi_\xi = 4i\bar{\xi}, \end{aligned} \quad (3.9)$$

and introduce the canonical Poisson brackets

$$\{\lambda, p_\lambda\} = \{\phi, p_\phi\} = 1, \quad \{\psi, \pi_\psi\} = \{\xi, \pi_\xi\} = -1. \quad (3.10)$$

From the explicit form of the fermionic momenta (3.10) it follows that we have second-class constraints. In order to resolve them, we will pass to the Dirac brackets for the canonical variables<sup>3</sup>

$$\begin{aligned} \{\lambda, \tilde{p}_\lambda\} &= 1, \quad \{\bar{\lambda}, \tilde{p}_{\bar{\lambda}}\} = 1, \quad \{\psi, \bar{\psi}\} = \frac{i}{8}, \quad \{\xi, \bar{\xi}\} = \frac{i}{8} \\ \{p_\phi, \tilde{p}_\lambda\} &= 2mh_{\phi\phi} \frac{\bar{\lambda}}{1+\lambda\bar{\lambda}} + 2imh_{\phi\lambda}, \quad \{p_\phi, \tilde{p}_{\bar{\lambda}}\} = 2mh_{\phi\phi} \frac{\lambda}{1+\lambda\bar{\lambda}} - 2imh_{\phi\bar{\lambda}}, \\ \{\tilde{p}_\lambda, \tilde{p}_{\bar{\lambda}}\} &= -2im \left( h_{\lambda\bar{\lambda}} - \frac{h_{\phi\phi}}{1+\lambda\bar{\lambda}} \right), \end{aligned} \quad (3.11)$$

where the bosonic momenta  $(\tilde{p}_\lambda, \tilde{p}_{\bar{\lambda}})$  have been defined as

$$\begin{aligned} \tilde{p}_\lambda &= p_\lambda - mA_\lambda, \quad A_\lambda = 2h_\phi \frac{\bar{\lambda}}{1+\lambda\bar{\lambda}} + 2ih_\lambda, \\ \tilde{p}_{\bar{\lambda}} &= p_{\bar{\lambda}} - mA_{\bar{\lambda}}, \quad A_{\bar{\lambda}} = 2h_\phi \frac{\lambda}{1+\lambda\bar{\lambda}} - 2ih_{\bar{\lambda}} \end{aligned} \quad (3.12)$$

Now, one may check that the following supercharges:

$$\begin{aligned} Q_1 &= (\tilde{p}_\phi + i\lambda\tilde{p}_\lambda)(\xi + \lambda\psi) + i\tilde{p}_{\bar{\lambda}}(\psi - \bar{\lambda}\xi) + 8\psi\bar{\psi}\xi + 2imh_\phi\xi, \\ Q_2 &= (\tilde{p}_\phi - i\bar{\lambda}\tilde{p}_{\bar{\lambda}})(\psi - \bar{\lambda}\xi) + i\tilde{p}_\lambda(\xi + \lambda\psi) - 8\psi\xi\bar{\xi} - 2imh_\phi\psi, \\ \bar{Q}^1 &= (\tilde{p}_\phi - i\bar{\lambda}\tilde{p}_{\bar{\lambda}})(\bar{\xi} + \bar{\lambda}\bar{\psi}) - i\tilde{p}_\lambda(\bar{\psi} - \lambda\bar{\xi}) + 8\psi\bar{\psi}\bar{\xi} - 2imh_\phi\bar{\xi}, \\ \bar{Q}^2 &= (\tilde{p}_\phi + i\lambda\tilde{p}_\lambda)(\bar{\psi} - \lambda\bar{\xi}) - i\tilde{p}_{\bar{\lambda}}(\bar{\xi} + \bar{\lambda}\bar{\psi}) - 8\bar{\psi}\xi\bar{\xi} + 2imh_\phi\bar{\psi} \end{aligned} \quad (3.13)$$

and the Hamiltonian

$$\begin{aligned} H &= \frac{(1+\lambda\bar{\lambda})^2}{4} \left( \tilde{p}_\lambda - i\frac{\bar{\lambda}p_\phi}{1+\lambda\bar{\lambda}} \right) \left( \tilde{p}_{\bar{\lambda}} + i\frac{\lambda p_\phi}{1+\lambda\bar{\lambda}} \right) + \frac{1}{4}p_\phi^2 + m^2h_\phi^2 \\ &\quad + 8m(1+\lambda\bar{\lambda}) \left[ h_{\lambda\bar{\lambda}}(\bar{\xi}\xi - \bar{\psi}\psi) + (ih_{\phi\lambda} + \bar{\lambda}h_{\lambda\bar{\lambda}})\bar{\psi}\xi + (-ih_{\phi\bar{\lambda}} + \lambda h_{\lambda\bar{\lambda}})\bar{\xi}\psi \right] \end{aligned} \quad (3.14)$$

form the standard  $N = 4$  superalgebra

$$\left\{ Q_i, \bar{Q}^j \right\} = \frac{i}{2}\delta_i^j H, \quad \{Q_i, Q_j\} = \{\bar{Q}^i, \bar{Q}^j\} = 0. \quad (3.15)$$

With this, we completed the classical description of  $N = 4$  supersymmetric mechanics on the sphere  $S^3$ . The corresponding Hamiltonian and supercharges are defined by (3.13) and (3.14). The freedom to choose the proper potential terms is hidden in one arbitrary function  $h$  obeying the Laplace equation on the  $S^3$  (2.9). Next, we analyze some specific interesting cases for the potential terms.

## 4 Potentials

The potential terms in the Hamiltonian (3.14) are completely defined by the function  $h$  obeying (2.9). Clearly, the most interesting potentials have to possess some additional symmetries. In this respect, the spherical symmetry of the solution seems to be the most important case. Let us firstly consider just such a type of potential.

<sup>3</sup>From now on, the symbol  $\{, \}$  stands for the Dirac brackets.

## 4.1 Spherically symmetric potential

The spherical symmetry, being rather hidden in stereographic coordinates, is quite evident in conformally flat coordinates (3.7). Remembering the relations between the stereographic coordinates  $\lambda, \bar{\lambda}, \phi$  and the conformally flat ones (3.7), one may easily find that the spherically symmetric case corresponds to a function  $h$  which depends only on the radius of  $S^3$  - the coordinate  $y$ :

$$y = \frac{e^{i\frac{\phi}{2}}\bar{\lambda} + e^{-i\frac{\phi}{2}}\lambda}{\sqrt{1+\lambda\bar{\lambda}}} = 2\frac{\mathbf{x}^2 - 1}{\mathbf{x}^2 + 1}. \quad (4.1)$$

Let us remind that the potential term is defined in terms of  $h_\phi$ , which also obeys the Laplace equation (2.9). This means that we have to pick up for  $h$  that solution which will give us the spherically symmetric  $h_\phi$ . It is rather easy to find that the proper solution is

$$h_\phi = a - 2b\frac{y}{\sqrt{4-y^2}}, \quad (4.2)$$

where  $a$  and  $b$  are arbitrary constants. The other derivatives of the function  $h$  which appear in the supersymmetric Hamiltonian (3.14) and Dirac brackets (3.11) are

$$\begin{aligned} h_{\phi\phi} &= \frac{4ibe^{i\frac{\phi}{2}}(1+\lambda\bar{\lambda})(\bar{\lambda} - e^{-i\phi}\lambda)}{(4 - e^{-i\phi}(\lambda - e^{i\phi}\bar{\lambda})^2)^{3/2}}, \quad h_{\lambda\bar{\lambda}} = \frac{4ibe^{i\frac{\phi}{2}}(\bar{\lambda} - e^{-i\phi}\lambda)}{(4 - e^{-i\phi}(\lambda - e^{i\phi}\bar{\lambda})^2)^{3/2}}, \\ h_{\phi\lambda} &= \frac{4be^{-i\frac{\phi}{2}}(2 + \bar{\lambda}(\lambda - e^{i\phi}\bar{\lambda}))}{(4 - e^{-i\phi}(\lambda - e^{i\phi}\bar{\lambda})^2)^{3/2}}, \quad h_{\phi\bar{\lambda}} = \frac{4be^{i\frac{\phi}{2}}(2 + \lambda(\bar{\lambda} - e^{-i\phi}\lambda))}{(4 - e^{-i\phi}(\lambda - e^{i\phi}\bar{\lambda})^2)^{3/2}}. \end{aligned} \quad (4.3)$$

When rewritten in conformally flat coordinates, these expressions read

$$\begin{aligned} h_\phi &= a - b\frac{1 - \mathbf{x}^2}{|\mathbf{x}|}, \quad h_{\phi\phi} = -\frac{b(1 + \mathbf{x}^2)^2 x_3}{4|\mathbf{x}|^3}, \quad h_{\lambda\bar{\lambda}} = -\frac{b(x_1^2 + x_2^2)x_3}{|\mathbf{x}|^3}, \\ h_{\phi\lambda} &= -i\frac{b(x_1 + ix_2)(2\mathbf{x}^2 - i(1 - \mathbf{x}^2)x_3)}{2|\mathbf{x}|^3}, \quad h_{\phi\bar{\lambda}} = i\frac{b(x_1 - ix_2)(2\mathbf{x}^2 + i(1 - \mathbf{x}^2)x_3)}{2|\mathbf{x}|^3}. \end{aligned} \quad (4.4)$$

Therefore, the Hamiltonian (3.14) in the case of spherically symmetric potentials reads

$$\begin{aligned} \mathcal{H} &= \frac{(1 + \lambda\bar{\lambda})^2}{4} \left( \tilde{p}_\lambda - i\frac{\bar{\lambda}p_\phi}{1 + \lambda\bar{\lambda}} \right) \left( \tilde{p}_{\bar{\lambda}} + i\frac{\lambda p_\phi}{1 + \lambda\bar{\lambda}} \right) + \frac{1}{4}p_\phi^2 + m^2 \left( a - b\frac{1 - \mathbf{x}^2}{|\mathbf{x}|} \right)^2 + 2mb(1 + \mathbf{x}^2)^2 \frac{\mathbf{x}}{|\mathbf{x}|^3} (\bar{\chi}\sigma\chi) = \\ &= \frac{(1 + \mathbf{x}^2)^2}{4} (\mathbf{p} - \mathbf{A})^2 + m^2 \left( a - b\frac{1 - \mathbf{x}^2}{|\mathbf{x}|} \right)^2 + 2mb(1 + \mathbf{x}^2)^2 \frac{\mathbf{x}}{|\mathbf{x}|^3} (\bar{\chi}\sigma\chi), \end{aligned} \quad (4.5)$$

where we combined the fermions  $\psi, \bar{\psi}, \xi, \bar{\xi}$  into the  $SU(2)$  spinor  $\chi = \begin{pmatrix} \psi \\ \xi \end{pmatrix}$ , with  $\sigma_i, i = 1, 2, 3$  being Pauli matrices.

Let us stress that the Hamiltonian (4.5) is just a particular case of the Hamiltonian (3.14), when the potential is chosen to be spherically symmetric and we partly use the coordinates (3.7). Therefore, it also appears in the anticommutators of the supercharges (3.13), as it occurs also for the Hamiltonian (3.14).

As it was argued in [3], the Hamiltonian of the MICZ-Kepler system on an arbitrary three-dimensional space with  $so(3)$ -invariant conformally flat metric  $ds^2 = G(r)(dx_1^2 + dx_2^2 + dx_3^2)$  should have the form

$$\mathcal{H} = \frac{1}{2G(r)} (\mathbf{p} - e\mathbf{A}_g)^2 + \frac{e^2(g\phi)^2}{2} - eq\phi, \quad \text{rot}\mathbf{A}_g = -g \text{grad}\phi, \quad (4.6)$$

where the Coulomb potential  $\phi$ , which is the  $so(3)$ -invariant solution of the Laplace equation

$$\frac{\partial}{\partial x^i} \left( G^{1/2} \frac{x^i}{r} \frac{d\phi}{dr} \right) = 0, \quad (4.7)$$

reads

$$\phi = a + b \int \frac{dr}{r^2 \sqrt{G(r)}}, \quad (4.8)$$

with  $a$  and  $b$  denoting arbitrary constants. In the case of the sphere  $S^3$  with  $G(r) = \frac{4}{(1+r^2)^2}$ , one may immediately conclude that the Coulomb potential has the form given by the first equation in (4.4). Thus, the bosonic part of the Hamiltonian (4.5) does completely coincide with the Hamiltonian of the charged particle on the sphere  $S^3$  moving in the field of Dirac dyon, whereas the fermionic part is just the Zeeman energy,  $U_Z = -\mathbf{B}\mathbf{M}$ , i.e. the energy of the interaction between the particle magnetic moment  $\mathbf{M} = 8e(\bar{\chi}\sigma\chi)$  and the magnetic field of the dyon, which has the monopole-like nature  $\mathbf{B} = g\frac{1}{G(x)}\frac{\mathbf{x}}{x^3} = g\frac{(1+\mathbf{x}^2)^2}{4}\frac{\mathbf{x}}{x^3}$ . Thus, one should identify  $b$  with magnetic charge of the dyon  $g$  and  $m$  - with the electric charge of the moving particle  $e$ . Moreover, in order to obtain proper Coulomb potential term corresponding to the interaction between moving particle and electric charge of the dyon  $eq\phi$  one should put  $a = \frac{eq}{2g}$ :

$$\begin{aligned}\mathbf{H} &= \frac{(1+\mathbf{x}^2)^2}{4}\mathbf{p}^2 + e^2(a + g\phi)^2 + \mathbf{B}\mathbf{M}, \\ \phi &= \frac{1-\mathbf{x}^2}{x}, \quad \mathbf{B} = g\frac{(1+\mathbf{x}^2)^2}{4}\frac{\mathbf{x}}{x^3}\end{aligned}\tag{4.9}$$

Thus, we conclude that the Hamiltonian (4.5) describes the  $N = 4$  supersymmetric MICZ–Kepler system on  $S^3$ . Of course, one may include into the Hamiltonian an arbitrary number of monopoles (4.4), in full analogy with the flat case [2]. We would like to stress that, while in the fermionic sector all terms coming from different monopoles will just sum up, the corresponding bosonic potential will be the square of the sum. So, additional cross-terms will appear. These terms are definitely needed, in order to have  $N = 4$  supersymmetry. Moreover, in a full analogy with the flat case, just this structure of the potential seems to be absolutely necessary for the integrability of the model, at least in the two monopoles case.

## 4.2 Cylindrically symmetric potential

It is clear that the stereographic coordinates are not so suitable to describe the spherically symmetric solutions of the Laplace equation on  $S^3$ . The “radial” variables  $y$  (4.1) look rather artificial in stereographic coordinates. Moreover, when analyzing the structure of (4.1) one may wonder whether the similar combination  $y_3$

$$y_3 = i\frac{e^{i\frac{\phi}{2}}\bar{\lambda} - e^{-i\frac{\phi}{2}}\lambda}{\sqrt{1+\lambda\bar{\lambda}}},\tag{4.10}$$

is suitable to get the particular solution of the Laplace equation. Indeed, it turns out that this is precisely the case. The corresponding solution has the same form as (4.2):

$$\tilde{h}_\phi = a_1 - 2b_1\frac{y_3}{\sqrt{4-y_3^2}}.\tag{4.11}$$

Passing to conformally flat coordinates, we get

$$\tilde{h}_\phi = a_1 - \frac{2b_1x_3}{\sqrt{(1+\mathbf{x}^2)^2 - 4x_3^2}}.\tag{4.12}$$

The remaining needed functions appearing in the Hamiltonian can be easily found from (4.11). At any rate, the explicit form of the potential (4.12) yields us informations about the cylindrical symmetry (for rotations around the  $x_3$  axis) of the solution.

It is worth to notice that the similar cylindrically symmetric solutions, with  $x_3$  being replaced by  $x_1$  and  $x_2$ , follow from two other solutions of the Laplace equations. They have the same form as (4.11), with the replacements  $y_3 \rightarrow y_1$  and  $y_3 \rightarrow y_2$ , where

$$y_2 = \frac{e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}}}{\sqrt{1+\lambda\bar{\lambda}}}, \quad y_1 = i\frac{e^{i\frac{\phi}{2}} - e^{-i\frac{\phi}{2}}}{\sqrt{1+\lambda\bar{\lambda}}}.\tag{4.13}$$

Finally, let us note that one may freely combine an arbitrary number of spherically symmetric monopoles with an arbitrary number of cylindrically symmetric ones, situated at arbitrary points. Moreover, as it is completely clear from the form of the  $S^3$  Laplace in stereographic coordinates (2.9), one may generate a new solution from the known ones by differentiating/integrating the latter over  $\phi$ . In this way one may produce a series of solutions originating from spherical/cylindrical symmetric monopoles. Of course, in order to decide which ones among them are really interesting, one should involve either physical arguments or integrability properties.



## 5 Conclusion

In this paper we derived the Hamiltonian and supercharges of the  $N = 4$  supersymmetric MICZ-Kepler system on  $S^3$ . We found the proper potential terms with spherical and cylindrical symmetry. In the case of spherically symmetric potential, we explicitly showed that in the bosonic sector our Hamiltonian describes the motion of the probe particle on the sphere  $S^3$  in the field of  $n$  Dirac dyons sitting at arbitrary points. The structure of the potential terms in the multi-center cases is very similar to the “flat” MICZ-Kepler system [2]. It is quite important that, while in the fermionic sector all terms coming from different monopoles will just be summing up, the corresponding bosonic potential will be the square of the sum. So, additional cross-terms will appear. These cross-terms are quite necessary for having  $N = 4$  supersymmetry.

One of the most interesting immediate problems is to analyze the integrability properties of the constructed system. We expect that, at least the two dyons system, will correspond to an integrable case. Another intriguing question concerns the integrability of the cylindrically symmetric potentials. Finally, the very simple structure of the  $N = 4$  supersymmetrization of the particle on  $S^3$  raises the question of the existence of its  $N = 8$  superextensions. Unfortunately, at present, no known example exists for  $N = 8$  supersymmetric systems on constant curvature bosonic manifolds. Our results, presented in this work, show that the relevant  $N = 8$  supermultiplet, if it exists, should correspond to some extension of the nonlinear  $N = 4$  supermultiplet. The corresponding construction is rather involved. Moreover, the structure of the possible potential terms is much more restricted in the case of  $N = 8$  supersymmetry. We are hoping to report the corresponding results elsewhere.

Finally, we would like to comment the question raised in [3]: whether it is possible to construct  $N = 4$  supersymmetric mechanics in which function describing the potential term obeys the same equation as metrics in the bosonic kinetic terms did. In the present paper we demonstrated that such situation indeed realized in the case of the sphere  $S^3$ . But the main ingredient we used was the nonlinear  $N = 4$  supermultiplet intrinsically related with  $S^3$  [7]. Now we do not know another  $N = 4$  supermultiplets with three physical bosonic components, beside linear tensor and nonlinear ones. So, the construction of a such  $N = 4$  supersymmetric mechanics seems to be a rather problematic.

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